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Generalized Dimensions, Entropies, and Liapunov Exponents from the Pressure Function for Strange Sets

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For conformal mixing repellers such as Julia sets and nonlinear one-dimensional Cantor sets, we connect the pressure of a smooth transformation on the repeller with its generalized dimensions, entropies, and Liapunov exponents computed with respect to a set of equilibrium Gibbs measures. This allows us to compute the pressure by means of simple numerical algorithms. Our results are then extended to axiom-A attractors and to a nonhyperbolic invariant set of the line. In this last case, we show that a first-order phase transition appears in the pressure.

KEY WORDS: Thermodynamic formalism; topological pressure, generalized Liapunov exponents; generalized dimensions; Renyi entropies; repellers; strange attractors.

1. INTRODUCTION

The thermodynamic description of dynamical systems is a powerful method for investigating chaotic behavior. A relation between the free energy and the generalized dimensions was recently established[1,2] by means of this formalism. We want to extend these results and show how to compute the pressure by simple numerical algorithms.

We start by focusing our analysis on the one-dimensional linear Cantor set, since it is perhaps the simplest nontrivial dynamical system for

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which one can exactly compute all the relevant dynamical, geometrical, and thermodynamic variables. Moreover, an approximation scheme[3,4] allows one to extend all the results valid for the linear Cantor set to expanding repellers. The method is constructive and its convergence properties have been proved. It can thus be used for computations with the desired accuracy in nonlinear repellers.

In this paper we compute, for a large class of expanding maps of the interval, the generalized Renyi dimensions and entropies as well as the generalized Liapunov exponents (with respect to some remarkable ergodic measures) and show that they are directly related to the pressure function. We also extend our methods to repellers in dimension larger than one and to strange attractors.

Using the thermodynamic formalism (see, e.g., Refs. 5–7) we give the analytic expression of the pressure for a nonhyperbolic repeller, the map $T(x)=x^2-2$ defined on the interval $[-2, 2]$. In this case the pressure exhibits a phase transition in agreement with a previous result obtained by Bohr and Rand.[8]

For several other models, such as the quadratic maps with totally disconnected Julia sets, the Baker transformation, and the Hénon and Zaslavskii maps, the pressure is computed analytically or numerically and the corresponding values for the generalized dimensions, entropies, and Liapounov exponents are given.

2. PRESSURE AND GENERALIZED DYNAMICAL VARIABLES: DEFINITIONS

2.1. Pressure

The pressure of a continuous function φ with respect to a continuous transformation T on the totally T-invariant compact set J is determined by the variational principle[7]:

$$P(T, \varphi)=\sup_{\mu\in M_T(J)}\left(K(\mu)+\int_J \varphi(x)\,d\mu(x)\right) \tag{2.1}$$

where $M_T(J)$ is the set of the T-invariant probability measures μ on J and $K(\mu)$ is the Kolmogorov entropy of μ. If J is a mixing repeller, the supremum is achieved for a unique ergodic measure, called the "equilibrium Gibbs" measure for φ. In the following we only consider the pressure of the function

$$\varphi(x)=-\beta\log\|D_xT\| \qquad \text{with} \quad \beta\in\mathbb{R} \tag{2.2}$$

where $\|D_x T\|$ is the norm of the tangent map of the transformation at x (in a suitable metric on J) and we write for convenience

$$P(\beta) = P(T, -\beta \log \|D_x T\|) \tag{2.3}$$

When $\varphi(x)$ has the form (2.2), the topological definition of the pressure[7] becomes rather simple [the variational principle (2.1) follows from this definition]. In fact, let $\mathscr{A}^{(0)}$ be an open cover of the invariant set J and $\delta(\mathscr{A}^{(0)})$ be the diameter of the covering. We then define

$$Z_n(\beta, \mathscr{A}^{(0)}) = \inf_{\alpha} \left\{ \sum_{A \in \alpha} \sup_{x \in A} \|D_x T^n\|^{-\beta} \right.$$
$$\left. \text{with } \alpha \text{ finite subcover of } \mathscr{A}^{(n-1)} \right\} \tag{2.4}$$

where $\mathscr{A}^{(n-1)}$ denotes the dynamical cover:

$$\mathscr{A}^{(n-1)} = \bigvee_{i=0}^{n-1} T^{-i} \mathscr{A}^{(0)} \tag{2.5}$$

It is possible to prove the existence of the following limit:

$$P(\beta, \mathscr{A}^{(0)}) = \lim_{n \to \infty} \frac{1}{n} \log Z_n(\beta, \mathscr{A}^{(0)}) \tag{2.6}$$

and the pressure of the function (2.2) can be written as

$$P(\beta) = \lim_{\varepsilon \to 0} (\sup P(\beta, \mathscr{A}^{(0)}) \colon \delta(\mathscr{A}^{(0)}) < \varepsilon) \tag{2.7}$$

We use this definition in Appendix B in order to compute the pressure of the Baker transformation.

2.2. Generalized Dimensions

The generalized dimensions[9] are a useful tool for describing the singularity spectrum of a measure. Let $\mathscr{A}^{(0)}$ be a cover of the set J by closed sets of small diameter that intersect only in the boundaries (a "partition" of J with a nonrigorous terminology), such that the dynamical partition $\mathscr{A}^{(n)}$ [see (2.5)] has a diameter which vanishes in the limit $n \to \infty$ (generating partition). We can thus define the partition function for real q and τ:

$$H_J(q, \tau, n) = \sum_{A_\alpha^{(n)} \in \mathscr{A}^{(n)}} \frac{\mu(A_\alpha^{(n)})^q}{\delta(A_\alpha^{(n)})^\tau} \tag{2.8}$$

where $A_\alpha^{(n)}$ is an element of $\mathscr{A}^{(n)}$, $\delta(A) = \operatorname{diam} A$ and μ is a T-invariant Borel probability measure on J. If for fixed q and $n \to \infty$ there exists a "changeover point" $\tau_\mu(q)$ such that for $\tau < \tau_\mu(q)$, $H_J \to 0$ and for $\tau > \tau_\mu(q)$, $H_J \to \infty$, we call

$$D_\mu(q) = \tau_\mu(q)/(q-1) \tag{2.9}$$

the generalized dimension of order q of the set J. These dimensions obviously depend on the measure μ (see, e.g., Ref. 4) and, in general, also on the partition $\mathscr{A}^{(0)}$.

2.3. Renyi Entropies

The Renyi entropies[10] are defined with regard to a T-invariant measure μ on the set J. If $\mathscr{A}^{(0)}$ is a μ-measurable generating partition, let us define the q-order entropy of the partition $\mathscr{A}^{(n)}$ as

$$\mathscr{F}_\mu(q, n) = \frac{1}{1-q} \log \sum_{A_\alpha^{(n)} \in \mathscr{A}^{(n)}} \mu(A_\alpha^{(n)})^q \tag{2.10}$$

The Renyi entropies are then the thermodynamic limits of $\mathscr{F}$, which are independent of the partition $\mathscr{A}^{(0)}$ if it is generating:

$$h_\mu(q) = \lim_{n \to \infty} \frac{1}{n} \mathscr{F}_\mu(q, n) \tag{2.11}$$

The Kolmogorov entropy $K(\mu)$ is recovered in the limit $q \to 1$, while the topological entropy h_{TOP} is recovered in the limit $q \to 0$.

2.4. Generalized Liapunov Exponents

The generalized Liapunov exponents[11] have been introduced as an indicator of the intermittency degree in chaotic systems and can be computed by means of simple numerical algorithms if the evolution equations are explicitly known,[11,12] in contrast with the Renyi entropies.

Let us now define these exponents with regard to an invariant ergodic measure μ on J such as

$$L_\mu(q) = \lim_{n \to \infty} \sup \frac{1}{n} \log \int_J \|D_x T^n\|^q \, d\mu(x) \tag{2.12}$$

For expanding systems the characteristic maximal Liapunov exponent $\lambda(\mu)$ is given by the derivative of $L_\mu(q)$ at $q=0$:

$$\lambda(\mu)=\lim_{n\to\infty}\sup\frac{1}{n}\int_J \log\|D_x T^n\|\,d\mu(x)$$

$$=\left.\frac{dL_\mu}{dq}\right|_{q=0}=\lim_{q\to 0}\frac{L_\mu(q)}{q} \tag{2.13}$$

3. CONNECTION OF THE DYNAMICAL VARIABLES WITH THE PRESSURE

We first consider two classes of repellers, the linear and nonlinear Cantor sets, which provide natural partitions on which all the dynamical variables can be easily computed. We can thus establish the relations with the topological pressure in the case of two relevant measures: the "balanced" measure μ_B, which maximizes the Kolmogorov entropy [i.e., $K(\mu_B)=h_{\text{TOP}}=h_\mu(q=0)$, for all μ], and the "uniform" measure μ_U, which maximizes the Hausdorff dimension of the measure.[15] We shall show that all the Renyi entropies are equal to h_{TOP} for the "balanced" measure, while all the generalized dimensions are equal to D_{H}, the Hausdorff dimension of the support, for the "uniform" measure. In this sense, the "uniform" measure is not multifractal,[9] in contrast to the "balanced" measure. Let us now consider a map T from an open neighborhood $V\supset[0,1]$ into $\mathbb{R}$ of class C^2 on V with the following properties (see Fig. 1):

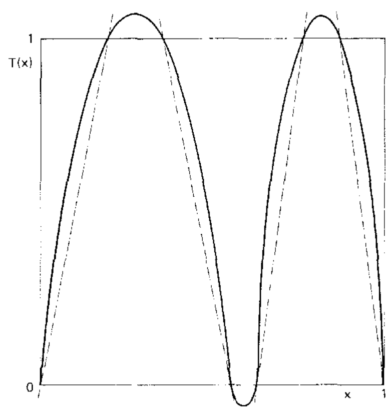

Fig. 1. Example of expanding map $T(x)$ of the interval $[0,1]$ with its piecewise linear approximation L_1.

(i) $T^{-1}([0,1]) \subset [0,1]$.

(ii) T is expanding on $T^{-1}([0,1])$, that is, $|T'(x)| \geqslant B > 1$ for all $x \in T^{-1}([0,1])$ [here and in the following $T'(x)$ also indicates the derivative of T with respect to x].

(iii) $T^{-1}([0,1]) = \bigcup_{k=1}^{s} I_k$, where I_k are closed, disjoint intervals $I_k \cap I_j = \varnothing$ for $k \neq j$.

The repeller J of the map

$$J = \bigcap_{n=0}^{\infty} T^{-n}([0,1]) \tag{3.1}$$

is an invariant Cantor set.

An important subclass is given by the maps $T(x)$ that are piecewise linear on $T^{-1}([0,1])$ (see Fig. 1). In this case we call the repeller J a linear Cantor set. Letting $T_k^{-1}(x)$ be the inverse of the restriction of $T(x)$ to the interval I_k where it is univalent, we shall write

$$T_k^{-1}(x) = a_k + \varepsilon_k \gamma_k x, \qquad \varepsilon_k = \pm 1, \qquad x \in [0,1]$$

and refer to the positive constants $\gamma_k < 1$ as scales of the linear Cantor J. We notice that $|T'(x)| = \gamma_k^{-1} > 1$ for $x \in I_k$, in agreement with (ii).

The partitions are easily constructed in this case: indeed, $\mathscr{A}^{(0)} = \bigcup_{k=1}^{s} (I_k \cap J)$, while

$$\mathscr{A}^{(n-1)} = \bigcup_{k_1,\ldots,k_n} A_{k_1,\ldots,k_n}, \qquad A_{k_1,\ldots,k_n} = [T_{k_n}^{-1} \cdots T_{k_1}^{-1}([0,1])] \cap J \tag{3.2}$$

with $k_J = 1,\ldots, s$ for $J = 1,\ldots, n$. We observe that

$$A_{k_1,\ldots,k_n} = T_{k_n}^{-1}(A_{k_1,\ldots,k_{n-1}}) \subset T_{k_n}^{-1}([0,1]) = I_{k_n}$$

As a consequence we have, for $x \in A_{k_1,\ldots,k_n}$,

$$\|DT(x)\| = \gamma_{k_n}^{-1}, \qquad \|DT^n(x)\| = \gamma_{k_1}^{-1} \cdot \cdots \cdot \gamma_{k_n}^{-1} \tag{3.3}$$

This observation allows us to obtain the pressure of the function (2.2) restricted to the Cantor set J. In fact, choosing the sets A in (2.4) as $A_{k_1,\ldots,k_n} \in \mathscr{A}^{(n-1)}$, we have

$$Z_n(\beta) = \sum_{k_1,\ldots,k_n} (\gamma_{k_1}^{-1} \cdot \cdots \cdot \gamma_{k_n}^{-1})^{-\beta} = (\gamma_1^\beta + \cdots + \gamma_s^\beta)^n \tag{3.4}$$

and finally

$$P(\beta) = \log(\gamma_1^\beta + \cdots + \gamma_s^\beta) \tag{3.5}$$

We can introduce on the linear Cantor set a family of T-ergodic probability measures μ,[14] with the properties

$$\mu(I_k)=p_k, \qquad \sum_{k=1}^{s} p_k=1 \tag{3.6}$$

$$\mu(A_{k_n,\ldots,k_1})=p_{k_1}p_{k_2}\cdots p_{k_n} \tag{3.7}$$

The constants p_k are the weights of the measure.

Two measures are particularly interesting: the balanced measure μ_B, whose weights are equal ($p_1=p_2=\cdots=p_s=s^{-1}$), and the Gibbs "uniform" measure μ_U, whose weights are $p_k=\lambda_k^{D_H}$, where D_H is the Hausdorff dimension of J.

It is also possible to consider a balanced measure for the nonlinear Cantor sets defined by

$$\mu_B(A)=\frac{1}{s}\mu_B(TA) \tag{3.8}$$

where A is any measurable subset of J where T is injective. Now putting in the partition functions of Section 2 a measure μ with weights $p_1,\ldots, p_s$ and repeating the straightforward combinatorial arguments that lead to (3.4), it is simple to compute the generalized variables for the nonlinear Cantor sets. Indeed, the generalized dimension $D_\mu(q)$ satisfies the relation[9]

$$\sum_{i=1}^{s} p_i^q\gamma_i^{-D_\mu(q)(q-1)}=1 \tag{3.9}$$

while the Renyi entropies and the generalized Liapunov exponents are

$$h_\mu(q)(1-q)=\log(p_1^q+\cdots p_s^q) \tag{3.10}$$

$$L_\mu(q)=\log(p_1\gamma_1^{-q}+\cdots+p_s\gamma_s^{-q}) \tag{3.11}$$

For the balanced measure we thus obtain by comparing (3.5), (3.9), and (3.11)

$$P[-D_{\mu_B}(q)(q-1)]=qP(0) \tag{3.12}$$

$$L_{\mu_B}(q)=P(-q)-P(0) \tag{3.13}$$

beyond the trivial relation $h_{\mu_B}(q)=\ln s=P(0)$, for all q's. On the other hand, for the Gibbs "uniform" measure $p_k=\lambda_k^{DH}$ one has

$$h_{\mu_U}(q)(1-q)=P(q\cdot D_{\mathrm{H}}) \tag{3.14}$$

$$L_{\mu_U}(q)=P(D_{\mathrm{H}}-q) \tag{3.15}$$

beyond $D_{\mu_U}(q)=D_{\mathrm{H}}$ for all q's, which directly follows from (3.9).

3.1. Dynamical Variables for Nonlinear Cantor Set

All the preceding relations can be extended to the invariant Cantor sets J of nonlinear expanding maps T by approximating T by a sequence of piecewise linear maps.[3,4] Let us in fact consider the nth level of dissociation of the unit interval:

$$\gamma_{n,\alpha} = \delta(A_\alpha^{n-1}), \qquad \alpha = 1,..., s^n \tag{3.16}$$

where A_α^{n-1} is the αth element of the partition $\mathscr{A}^{(n-1)} = T^{-(n-1)}\mathscr{A}^{(0)}$, with $\mathscr{A}^{(0)} = T^{-1}[0, 1] \cap J$. The $\gamma_{n,\alpha}$ can be identified with the scales of a linear endomorphism whose invariant set is C_n, and the pressure P_n restricted to C_n is just given by (3.5). It has been shown[3,4] that the Hausdorff distance[35] between C_n and J vanishes when $n \to +\infty$ and that the pressure $P(\beta)$ with respect the mapping T is

$$P(\beta) = \lim_{n \to +\infty} \frac{1}{n} \log \sum_{A_\alpha^{(n-1)} \mathscr{A}^{(n-1)}} \delta(A_\alpha^{n-1})^\beta = \lim_{n \to +\infty} \frac{1}{n} P_n(\beta) \tag{3.17}$$

This result can be also proved by means of the Ruelle–Perron–Frobenius operator.[36] In fact, for an expanding one-dimensional map and more generally for a conformal mixing repeller[15] one has uniformly in $x \in J$:

$$P(\beta) = \lim_{n \to +\infty} \frac{1}{n} \log \sum_{y \in T_x^{-n}} \frac{1}{|(T^n)'(y)|^\beta} \tag{3.18}$$

Any preimage y of order n of x belongs to a different element $A_\alpha^{(n-1)}$ of $T^{-n}[0, 1]$ and the derivative $|(T^n)'(y)|^\beta$ is uniformly bounded, with respect to n, by the inverse of the diameter of $A_\alpha^{(n-1)}$. This follows from the identity (A4) in Appendix A and by the distortion argument used in the proof of Proposition 2 (see below). One thus recovers (3.17) in the thermodynamic limit. Let us note that (3.18) is quite useful if one wants to compute the pressure for connected repellers such as Julia sets.[37]

Using (3.17), we can compute the generalized dimensions with respect to the balanced measure. The partition function (2.8) in fact becomes

$$\begin{aligned} H_J(q, \tau, n) &= \sum_{A_\alpha^{(n)} \in \mathscr{A}^{(n)}} \frac{\mu_B(A_\alpha^{(n)})^q}{\delta(A_\alpha^{(n)})^\tau} \\ &= \left(\frac{1}{s}\right)^{q(n+1)} \sum_{A_\alpha^{(n)} \in \mathscr{A}^{(n)}} \delta(A_\alpha^{(n)})^{-\tau} \\ &= \exp\left\{(n+1)\left[-q \log s + \frac{1}{n+1} \log \sum_{A_\alpha^{(n)} \in \mathscr{A}^{(n)}} \delta(A_\alpha^{(n)})^{-\tau}\right]\right\} \end{aligned} \tag{3.19}$$

It follows that in the limit $n \to +\infty$, $\tau_{\mu_B}(q)$ must satisfy the relation

$$P(-\tau_{\mu_B}(q)) = q \log s = q \cdot h_{\mathrm{TOP}} = q \cdot P(0) \tag{3.20}$$

which has been proved in Refs. 1 and 2. Let us now compute the Renyi entropies with respect to μ_B and to μ_U. In the first case one sees that

$$h_{\mu_B}(q) = \log s = h_{\mathrm{TOP}}, \qquad \forall q \in \mathbb{R} \tag{3.21}$$

since $\mu_B(A_\alpha^{(n)}) = s^{-(n+1)}$

We have to stress that the ("inverse temperature") coefficient β in (2.2) parametrizes a whole class of equilibrium measures, say μ_β, which realize the maximum in the variational principle (2.1). The value $\beta = D_{\mathrm{H}}$ picks up the "uniform" measure μ_U, since μ_U is equivalent to the D_{H}-Hausdorff measure on J,[15,16] while the infinite-temperature limit $\beta = 0$ picks up the maximum entropy measure μ_B. In Appendix A we prove the following relation for the "uniform" measure:

Proposition 1

$$h_{\mu_U}(q)(1-q) = P(q \cdot D_{\mathrm{H}}) \tag{3.22}$$

It is also simple to see that the generalized dimensions $D_{\mu_U}(q)$ are constant and equal to D_{H} by inserting the bound (A.5) into (2.10) and recalling the Bowen–Ruelle formula[15]

$$P(D_{\mathrm{H}}) = 0 \tag{3.23}$$

Let us recall that there is another remarkable measure for which it is not difficult to compute the Renyi entropies. It is the measure μ_{SBR} corresponding to $\beta = 1$, for which it has been shown that[4,8]

$$K(\mu_{\mathrm{SBR}}) - \lambda(\mu_{\mathrm{SBR}}) = -\rho \tag{3.24}$$

where ρ is the escape rate[18] and $\lambda(\mu)$ is the Liapunov exponent computed with respect to the measure μ. Bohr and Rand[8] have called this measure the "Sinai–Bowen–Ruelle" measure for repellers, on the basis of its formal analogy with the physical measure for attractors and proved that

$$h_{\mu_{\mathrm{SBR}}}(q)(1-q) = P(q) - qP(1) \tag{3.25}$$

We will get this relation as a particular case of Proposition 3 below.

Let us finally consider the generalized Liapunov exponents $L_\mu(q)$. For the systems we are studying the definition (2.12) is equivalent to the following.

Proposition 2

$$L_\mu(q) = \lim_{n\to+\infty} \sup \frac{1}{n} \log \sum_{A_\alpha^{(n)} \in \mathscr{A}^{(n)}} \delta(A_\alpha^{(n)})^{-q} \cdot \mu(A_\alpha^{(n)}) \tag{3.26}$$

Proof of Proposition 2. By considerations quite similar to those adopted in the proof of Proposition 1, it can be shown that there is a constant $G \geqslant 1$ such that for every pair of points $(x, y) \in A_\alpha^{(n)}$ and for each $n > 0$, one has (see also Ref. 19)

$$G^{-1} |(T^n)'(y)| \leqslant |(T^n)'(x)| \leqslant G |(T^n)'(y)| \tag{3.27}$$

Using (3.27) and the identity (A.4), we can bound the integral in (2.12), supposing without any restriction that $q > 0$:

$$\begin{aligned} G^{-1} \sum_{A_\alpha^{(n)} \in \mathscr{A}^{(n)}} \delta(A_\alpha^{(n)})^{-q} \mu(A_\alpha^{(n)}) &\leqslant \int_J |(T^{n+1})'(x)|^q \, d\mu \\ &\leqslant G \sum_{A_\alpha^{(n)} \in \mathscr{A}^{(n)}} \delta(A_\alpha^{(n)})^{-q} \mu(A_\alpha^{(n)}) \end{aligned} \tag{3.28}$$

and the equivalence between (3.26) and (2.12) immediately follows from (3.28). The definition (3.26) is rather useful when the measure of an atom $A_\alpha^{(n)}$ is known. In the case of the balanced measure $\mu_B(A_\alpha^{(n)}) = s^{-(n+1)}$, so that one sees

$$L_{\mu_B}(q) = P(-q) - \log s = P(-q) - P(0) \tag{3.29}$$

where $P(0) = h_{\text{TOP}}$ by the variational principle of the pressure.

In the case of the "uniform" measure μ_U, using the bounds (A.5) on the measure of an atom of the partition, one easily gets

$$L_\mu(q) = P(D_\text{H} - q) \tag{3.30}$$

which was conjectured in Ref. 11. Let us stress that for μ_U and μ_B, the existence of the limit (3.26) can be proved. Moreover, the relation between the pressure and the generalized Liapunov exponents is useful for reconstructing the pressure itself, since the numerical calculations of $L_\mu(q)$ (Ref. 11) are much more direct than those of either $D_\mu(q)$ (Ref. 20) or $h_\mu(q)$ (Ref. 21).

Up to now we have given the relations that link the generalized exponents to the topological pressure $P(\beta)$ for the equilibrium measures realizing the maximum in (2.1) with $\beta = 0$, 1, and D_H respectively. Indeed, these measures are the most commonly considered for their relevance in numerical experiments. It is, however, possible to extend those relations to every equilibrium measure corresponding to a real β. In the context of Walter's theory,[36] one can show that an equilibrium measure μ_β is equivalent, with continuous Radon–Nikodym derivative, to a probability measure ν_β such that

$$\nu_\beta(TA) = \int_A \frac{|T'(x)|^\beta}{\exp[-P(\beta)]} \, d\nu_\beta(x)$$

taking any measurable subset A of J where T is injective. This relation is similar to (A1) and thus, by using the same arguments as in the proof of Proposition 1, one can prove[38] the following result.

Proposition 3. For the expanding repellers, the generalized exponents computed with respect to the equilibrium measures μ_β are connected to the topological function $P(\beta)$ by

$$P[\beta q - D_{\mu_\beta}(q)(q-1)] = qP(\beta)$$
$$h_{\mu_\beta}(q)(1-q) = -qP(\beta) + P(q \cdot \beta)$$
$$L_{\mu_\beta}(q) = P(\beta - q) - P(\beta)$$

Proposition 3 is quite easy to prove[37] for the linear Cantor sets with scales $\gamma_1,..., \gamma_s$, since in this case the equilibrium measures become particular balanced measures whose weights are given by

$$p_k = \gamma_k^\beta \Big/ \sum_{k=1}^{s} \gamma_k^\beta, \qquad k = 1,..., s$$

[see (3.6)].

3.2. Generalized Thermodynamic Relations

It is worth stressing that a knowledge of the set of the $L_{\mu_\beta}(q)$ with respect to a particular equilibrium measure μ_β that satisfies the variational principle (2.1) with φ given by (2.2) is fully equivalent to a knowledge of the Liapunov exponents $\lambda_\beta \equiv \lambda(\mu_\beta)$ computed with respect to the whole class of equilibrium measures μ_β at varying β.

Indeed, one sees from (2.4) and (2.6) that the derivative of the pressure[4] gives the Liapunov exponents of the equilibrium Gibbs measures:

$$\lambda_\beta = -\left.\frac{dP(\beta)}{d\beta}\right|_{\beta=\beta} \tag{3.31}$$

[4] If T is real analytic, the pressure is an analytic function of β.[14]

and thus that λ_β is the analog of the internal energy as well as $-P(\beta)/\beta$ of the free energy in thermodynamics.

For instance, from (3.29) and (3.30), we get

$$\left.\frac{dL_{\mu_B}(q)}{dq}\right|_{q=\bar{q}} = \left.\frac{dP(-q)}{dq}\right|_{q=\bar{q}} = \lambda_{\beta=-\bar{q}} \tag{3.32}$$

and

$$\left.\frac{dL_{\mu_U}(q)}{dq}\right|_{q=\bar{q}} = \left.\frac{dP(D_{\mathrm{H}}-q)}{dq}\right|_{q=\bar{q}} = \lambda_{\beta=D_{\mathrm{H}}-\bar{q}} \tag{3.33}$$

which reduce to the trivial identity (2.13) in the limit $q \to 0$, since $\mu_{\beta=0} = \mu_B$ and $\mu_{\beta=D_{\mathrm{H}}} = \mu_U$.

We have thus seen that for a given Gibbs measure (i.e., for a given "temperature" β^{-1}), the temporal intermittency (the finite-time fluctuations of the chaoticity degree) characterized by $L_{\mu_\beta}(q)$ depends on the asymptotic behavior characterized by the whole set of λ_β's. This is in full analogy with thermodynamics, where the finite-volume fluctuations of the energy at a given temperature are connected to the diagram of the free energy as function of the temperature.[22]

We can also generalize the well-known formula for conformal mixing repellers[4,23,24]:

$$\lambda(\mu) = K(\mu)/D_I(\mu) \tag{3.34}$$

which holds for any ergodic measure on J and where $D_I(\mu)$ is the Hausdorff dimension of the measure μ (information dimension), that is, roughly speaking, the Hausdorff dimension of the smallest subset J of full μ-measure.[25]

Inserting (3.20) into (3.29), we thus obtain, for the balanced measure,

$$\frac{L_{\mu_B}(\tau_{\mu_B}(q))}{\tau_{\mu_B}(q)} = \frac{h_{\mu_B}(q)}{D_{\mu_B}(q)} = \frac{h_{\mathrm{TOP}}}{D_{\mu_B}(q)} \tag{3.35}$$

On inserting (3.22) into (3.30), we obtain for the uniform measure

$$\frac{L_{\mu_U}(-\tau_{\mu_U}(q))}{-\tau_{\mu_U}(q)} = \frac{h_{\mu_U}(q)}{D_{\mu_U}(q)} = \frac{h_{\mu_U}(q)}{D_{\mathrm{H}}} \tag{3.36}$$

In the limit $q \to 1$, relations (3.35) and (3.36) reduce to (3.34) since $\tau_\mu(1) = 0$, $h_\mu(1) = K(\mu)$, and $D_\mu(1) = D_I(\mu)$, while $\lim_{x\to 0} L_\mu(x)/x = \lambda(\mu)$, for every equilibrium measure μ.

Let us finish this section by giving some bounds on the generalized Liapunov exponents. The first trivial bound,

$$L_\mu(q) \geqslant q \cdot \lambda(\mu) \tag{3.37}$$

follows from the Jensen's inequality, while giving the upper bound 1 to the measure of an atom and recalling (3.18), we obtain from (3.26)

$$L_\mu(q) \leqslant P(-q) \tag{3.38}$$

4. EXTENSION TO MORE GENERAL SYSTEMS

The one-dimensional hyperbolic systems described in Section 3 have an important generalization in the so-called conformal mixing repellers: the hyperbolic Julia sets are well-known examples. Besides the expanding properties and the topological transitivity,[6] they have the additional property that the tangent map in every point of the invariant set behaves like a scalar time on isometry. It is possible to do a smooth ergodic theory for conformal repellers that live in Riemannian manifolds of any degree[15,4]; moreover, they could be connected sets and therefore not have a Cantorian structure. The role of the Cantorian partitions is taken, for a generic conformal mixing repeller, by the Markov partitions.[6] Using the Markov partitions to construct the partition functions defining the various generalized dynamical variables of Section 2, one can exactly reproduce all the relations with the pressure as seen in the case of Cantorian repellers. We refer to Ref. 4 for the derivations; moreover, these results are used in Section 5 and are applied to a connected Julia set, for which the pressure can be computed perturbatively. Some of the previous relations can also be applied to Axiom-A attractors. There already exists an extension of the Bowen–Ruelle formula (3.23), which gives the Hausdorff dimension of the intersection of the unstable (resp. stable) manifold of a point with the basic set of a C^2 Axiom-A diffeomorphism T of a surface.[26,27] For our purposes, we consider an Axiom-A attractor $\Lambda \subset \mathbb{R}^2$ such that the (locally smooth) unstable manifold $W^u(x)$ of a point $x \in \Lambda$ intersects the attractor in a Cantor set. We define by $\widehat{D_x T_U}$ the norm of the tangent map restricted to the unstable subspaces at the point $x \in \Lambda$; successively we put on Λ the physical measure μ_{PH} (Sinai–Bowen–Ruelle measure), which, as is well known, is smooth along unstable directions and can be reconstructed by a time series starting from almost all the points in the basin of attraction.[5] Using the properties of the Markov partitions for Axiom-A attractors,[5,6] it is possible to construct a partition $\mathscr{A}$ of Λ such that, if $A(x)$ is the element of $\mathscr{A}$ containing x, then $A(x) \subset W^U(x)$ and $A^{-1}(x) \subset A(x)$, where

$A^{-1}(x)$ is the element of $T^{-1}\mathscr{A}$ containing x (see Ref. 28 for more details). Roughly speaking, $T^{-n}\mathscr{A}$ dissects the unstable manifolds as the dynamical Markov partition does for a mixing repeller; carrying on this analogy and taking account of the smoothness of μ_{PH} on the unstable manifolds (which approximately means that $\bar{\mu}_{\mathrm{PH}}(A_\alpha^{(n)}) \sim \delta(A_\alpha^{(n)})$, where $\delta(A_\alpha^{(n)})$ is the diameter of $A_\alpha^{(n)} \in \bigvee_{k=0}^{n} T^{-k}\mathscr{A}$, and $\bar{\mu}_{\mathrm{PH}}$ is the conditional measure along $W^U(x)$], we can write two expressions parallel to Eqs. (3.30) and (3.22) for the repellers:

$$\begin{aligned} L_{\mu_{\mathrm{PH}}}(q) &= \limsup_{n\to+\infty} \frac{1}{n} \log \int_A \|DT^n(x)\|^q \, d\mu_{\mathrm{PH}} \\ &= \limsup_{n\to+\infty} \frac{1}{n} \log \int_A \widehat{D_x T_U^n} \, d\mu_{\mathrm{PH}} = P(1-q) \end{aligned} \tag{4.1}$$

where

$$D_x T_U^n = \prod_{j=0}^{n-1} D_{T_x^j} T_U; \qquad P(\beta) = P(T, -\beta \log \widehat{D_x T_U})$$

and

$$h_{\mu_{\mathrm{PH}}}(q) = \frac{1}{1-q} P(q) \tag{4.2}$$

We rigorously prove these formulas in Appendix B for the attracting set of the Baker transformation. However, the same arguments should work in general for hyperbolic attractors that are locally the Cartesian product of a Cantor set with an interval. Combining (4.1) and (4.2), we obtain the relation

$$L_{\mu_{\mathrm{PH}}}(1-q) = (1-q)\, h_{\mu_{\mathrm{PH}}}(q) \tag{4.3}$$

which corresponds to (3.36) with $D_{\mathrm{H}} = 1$ and $\tau_{\mu_U}(q) = (q-1)$.

If this is the case, a knowledge of the generalized Liapunov exponents is equivalent to that of the Renyi entropies, as recently conjectured by Paladin and Vulpiani[21] for some generic systems.

5. EXAMPLES

5.1. Repellers

Let us consider the nonlinear quadratic map

$$T(x) = x^2 - p, \qquad p > 2 \tag{5.1}$$

whose invariant Cantor set J is a totally disconnected Julia set of the line.[29] When p is large, it has been shown[3] that J is well approximated by a nonlinear Cantor set generated by a linear map with two equal scales γ:

$$\gamma = \frac{1}{2}\frac{(p-q)^{1/2}}{2q} \tag{5.2}$$

where $q = \frac{1}{2}[1 + (1+4p)^{1/2}]$. In this case, the pressure becomes

$$P(\beta) = \log 2 + \beta \log \gamma = \log 2 - \beta\lambda(\mu_B) \tag{5.3}$$

where

$$\lambda(\mu_B) = \log(2\sqrt{p})\left[1 + O\left(\frac{1}{p^2}\right)\right] \tag{5.4}$$

is the Liapunov exponent of the balanced measure μ_B.

When p approaches 2, more and more piecewise linear approximations are needed and numerical computations become essential. The pressure is then given by two straight lines, which can be fitted by

$$P(\beta) = \begin{cases} -2\beta \log 2 & \text{for} \quad \beta < -1 \\ -(\beta - 1)\log 2 & \text{for} \quad \beta > -1 \end{cases} \tag{5.5}$$

as shown in Fig. 2. A phase transition occurs at $\beta_C = -1$; this is a consequence of the fact that we are entering a nonhyperbolic region.

We shall return to these formulas in more detail in Section 6. Our results can be extended to connected repellers. Here we only consider an

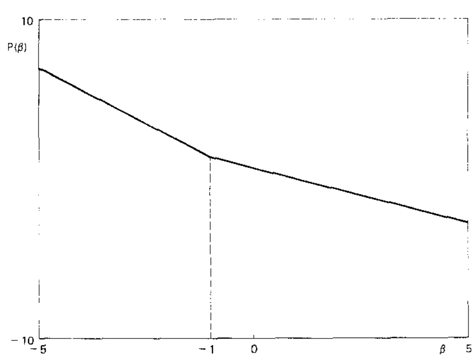

Fig. 2. Pressure $P(\beta)$ versus β for the nonhyperbolic map $x_{n+1} = x_n^2 - p$, $p = 2$. One can see that a phase transition occurs at $\beta_c = -1$.

example, given by the Julia set of the map $T(x) = x^2 - p$, $x \in \mathbb{C}$, for small p, which is a fractal Jordan curve homeomorphic to the unit circle.[29] By the perturbative calculation of the zeta function, Ruelle[15] computed the pressure at the second order in $|p|$, which reads

$$P(\beta) = \tfrac{1}{4}|p|^2 + \log 2 - \beta \log 2 + O(|p|^3) \tag{5.6}$$

From (3.20) and (3.29) and with respect to all the equilibrium measures, we obtain that, at the second order in $|p|$, the generalized dimensions are constant and the generalized Liapunov exponents are linear in q: We have, $\forall \beta$,

$$D_{\mu_\beta}(q) = 1 + \frac{|p|^2}{4 \log 2} + O(|p|^3) \tag{5.7}$$

$$L_{\mu_\beta}(q) = q \log 2 = q\lambda(\mu_B) \tag{5.8}$$

since, for small $|p|$, the μ_B-Liapunov exponent is log 2.[23] Let us stress that as far as we know this is the first perturbative calculation of the generalized exponents in the Julia set of (5.1). It would be interesting to know if a nontrivial dependence on q ("multifractality" following Parisi's terminology[9] appears at higher p orders as some numerical calculations seem to suggest.[37]

5.2. Strange Attractors

We want now to check if (4.3) holds in the case of generic attractors that have the physical measure, where there exist numerical methods to compute independently the generalized Liapunov exponents and the Renyi entropies.[11,20] Let us, for instance, consider the attractor of the Hénon map $(x_{n+1}, y_{n+1}) = (y_{n+1} + 1 - 1.4x_n^2, 0.3x_n)$, which is not an Axiom-A attractor.[31] By a numerical calculation we have estimated

$$\begin{aligned} L(q=1) &= 0.45 \\ L(q=-1) &= 0.40 \\ \lambda &= 0.4196 \end{aligned} \tag{5.9}$$

and we refer the reader to Benzi *et al.*[11] for a plot of $L(q)$ versus q. We have also done the same calculations for the Zaslavskii map

$$\begin{aligned} (x_{n+1}, y_{n+1}) = (&[x_n + v(1 + ry_n) + \varepsilon v r \cos 2\pi y_n] \bmod 1, \\ &e^{-r}[y_n + \varepsilon \cos 2\pi x_n]) \end{aligned}$$

with $r = (1 - e^{-\Gamma})/\Gamma$, $\Gamma = 3$, and $\nu = 400/3$, $\varepsilon = 0.3$:

$$\begin{aligned} L(q = 1) &= 3.80 \\ L(q = -1) &= 3.56 \\ \lambda &= 3.689 \end{aligned} \tag{5.10}$$

The values of $L(1)$ and λ are in good agreement with those for $h(0)$ and $h(1)$ obtained by Grassberger and Procaccia.[20] Nevertheless, there is a significant difference between $L(-1)$ and $h(2)$. We do not know if this discrepancy is due to the fact that the systems are not Axiom-A. Let us also note that a linear extrapolation of $h(2)$ from $h(0)$ and $h(1)$ gives values close to those obtained applying (4.3) to $L(-1)$ in both cases. This might suggest that in nonhyperbolic systems a phase transition in the pressure can occur and that relation (4.3) is not valid for all q values.

The Zaslavskii map can be regarded as a perturbation of the generalized Baker transformation, whose pressure $P(\beta)$ is real analytic. We therefore discuss in detail this simple mapping and its attracting set Λ.

5.3. Baker's Transformation: An Exactly Solved Toy Model

The Baker transformation T is defined on the unit square into itself and is given iteratively by

$$x_{n+1} = \begin{cases} \gamma_a x_n & \text{if } y_n < \alpha \\ \dfrac{1}{2} + \gamma_b x_n & \text{if } y_n > \alpha \end{cases}$$
$$y_{n+1} = \begin{cases} \dfrac{1}{\alpha} y_n & \text{if } y_n < \alpha \\ \dfrac{1}{1-\alpha}(y_n - \alpha) & \text{if } y_n > \alpha \end{cases} \tag{5.11}$$

with $0 \leqslant x_n, y_n \leqslant 1$, and $0 < \gamma_a < \gamma_b < 1/2$, $\alpha \leqslant 1/2$. It is well known[33] that Λ is the Cartesian product of a Cantor set and the vertical interval $[0, 1]$; if $\widehat{D_x T_U}$ is the Jacobian of the map along the unstable subspaces, we have the following two Propositions which we prove in Appendix B:

Proposition 4

$$P(q) \equiv P(T, -q \log \widehat{D_x T_U}) = \log[\alpha^q + (1-\alpha)^q] \tag{5.12}$$

From Proposition 4, we can obtain the following expressions for the generalized Liapunov exponents and entropies with respect to the physical measure μ_{PH}:

Proposition 5

$$L_{\mu_{\mathrm{PH}}}(q) = \log[\alpha^{1-q} + (1-\alpha)^{1-q}] \tag{5.13a}$$

and

$$h_{\mu_{\mathrm{PH}}}(q) = \frac{1}{1-q} P(q) = \frac{1}{1-q} \log[\alpha^q + (1-\alpha)^q] \tag{5.13b}$$

which justify (4.1) and (4.2) for Baker's transformation. Note that taking the limits $q \to 0$, $q \to 1$ in (5.13a), (5.13b), one gets

$$\lambda(\mu_{\mathrm{PH}}) = K(\mu_{\mathrm{PH}}) = \alpha \log\left(\frac{1}{\alpha}\right) + (1-\alpha) \log\left(\frac{1}{1-\alpha}\right) \tag{5.14}$$

which agrees with the exact values obtained by the direct application of the definitions.[20] If we know the conditional probabilities of the physical measure along the stable directions, we can try to compute the generalized dimensions of the (Cantorian) intersection of the attractor with the stable manifolds. In our case, we construct the partition function

$$H(q, \tau, n) = \sum_{A_\alpha^{(n)} \in \mathscr{A}^{(n)}} \frac{\bar{\mu}_{\mathrm{PH}}(A_\alpha^{(n)})^q}{\delta(A_\alpha^{(n)})^\tau} \tag{5.15}$$

where $\mathscr{A}^n = T^n([0, 1] \times [0, 1]) \cap [0, 1] \cap A$ is the intersection of the nth iterate of the unit square with (each) horizontal interval, and $\bar{\mu}_{\mathrm{PH}}(A)$ is the conditional measure of A along the stable directions.

These measures are done as follows[30]: if $A_\alpha^{(n)}$ is a generic atom of diameter $\delta(A_\alpha^{(n)}) = \prod_{k=1}^n \gamma_{\alpha_k}$, with $\alpha_k = a$ or b, then

$$\bar{\mu}_{\mathrm{PH}}(A_\alpha^{(n)}) = \prod_{k=1}^{n} p_{\alpha_k} \qquad \text{with} \quad p_a = \alpha,\ p_b = 1-\alpha \tag{5.16}$$

with $p_a = \alpha$ and $p_b = 1-\alpha$. This is obviously a particular kind of balanced measure for the one-dimensional Cantor set $\bigcap_{k=0}^\infty \mathscr{A}^{(k)}$. Then the partition function (5.15) becomes

$$\begin{aligned} H(q, \tau, n) &= \sum_{\alpha_1} \cdots \sum_{\alpha_n} \prod_{k=1}^{n} \gamma_{\alpha_k}^{-\tau} p_{\alpha_k}^q \\ &= [\alpha^q \gamma_a^{-\tau} + (1-\alpha)^q \gamma_b^{-\tau}]^n \end{aligned} \tag{5.17}$$

In order to have the partition function of order one when $n \to +\infty$, we put the expression in parentheses in (5.17) equal to 1: this defines the changeover point $\tau^s_{\mu_{\mathrm{PH}}}(q)$ corresponding to the transverse generalized dimension $D^s_{\mu_{\mathrm{PH}}}(q)$. Finally, remembering the smoothness of the attractor along the unstable manifolds, we can write the following expression for the generalized dimension $D_{\mu_{\mathrm{PH}}}(q) = D^s_{\mu_{\mathrm{PH}}}(q) + 1$ of the attractor itself:

$$\alpha^q \gamma_a^{[D_{\mu_{\mathrm{PH}}}(q) - 1](1-q)} + (1-\alpha)^q \gamma_b^{[D_{\mu_{\mathrm{PH}}}(q) - 1](1-q)} = 1 \tag{5.18}$$

which coincides with the calculation of Ref. 30.

6. AN EXAMPLE OF NONHYPERBOLIC SET

Let us finally discuss an example of a nonhyperbolic set by considering the map

$$\tilde{z} = T(z) = z^2 - p \qquad \text{with} \quad p = 2 \text{ and } z \in \mathbb{C} \tag{6.1}$$

In such a case we can compute exactly the Ruelle zeta function of the function $\varphi = \ln |T'|^{-\beta}$. Setting $z = 2\cos\vartheta$, $\tilde{z} = 2\cos\tilde{\vartheta}$, it follows that $T^n(z) = 2\cos(2^n\vartheta)$ as well as

$$|(T^n)'(z)| = 2^n \left| \frac{\sin(2^n\vartheta)}{\sin\vartheta} \right| \tag{6.2}$$

On the other hand, the fixed points of T^n are given by $2\cos\vartheta = 2\cos(2^n\vartheta)$. One therefore has

$$|T^{n'}(z)| \underset{z \in \mathrm{Fix}\, T^n}{\equiv} \begin{cases} 2^{2n} & \text{for} \quad \vartheta = 0 \\ 2^n & \text{for} \quad \vartheta \neq 0 \end{cases} \tag{6.3}$$

Let us introduce the Ruelle zeta function[6]:

$$\zeta(u) = \exp \sum_{n=1}^{\infty} (u^n/n)\, A_n$$

where

$$A_n = \sum_{x \in \mathrm{Fix}\, T^n} \exp(-\beta \ln |D_x T|)$$

It is known that $\zeta(u)$ is a meromorphic function of u for Axiom-A systems and that the location of the nearest pole to the origin is equal to the topological pressure $P(\beta)$. It results from (6.3) that

$$A_n \equiv \sum_{z \in \mathrm{Fix}\, T^n} |T^{n'}(z)|^{-\beta} = 2^{-2n\beta} + 2^{-n\beta}(2^n - 1) \tag{6.4}$$

and we thus get

$$\zeta(u) = -\frac{1}{2}\frac{(u-2^{\beta})}{(u-2^{2\beta})(u-2^{\beta-1})} \tag{6.5}$$

One easily checks that:

1. The nearest pole is either $2^{2\beta}$ or $2^{\beta-1}$, depending upon whether β is smaller or larger than -1. Defining, by analogy with the Axiom-A theory, the pressure by $\exp[-P(\beta)] =$ location of the nearest pole to the origin, one has

$$P(\beta) = \begin{cases} -2\beta \ln 2 & \text{if} \quad \beta < -1 \\ -(\beta-1)\ln 2 & \text{if} \quad \beta > -1 \end{cases} \tag{6.6}$$

2. The zero $u = 2^{\beta}$ is always outside the circle of center the origin containing the nearest pole.

In this case a transition occurs at $\beta_c = -1$, as already remarked[8,32] using different techniques. The presence of such a transition does not contradict the result of Ref. 15, since the set is not hyperbolic. The generalized dimensions can be obtained by extending formula (3.20):

$$D_{\mu_B}(q) = \begin{cases} 1 & \text{for} \quad q \leqslant 2 \\ q/[2(q-1)] & \text{for} \quad q > 2 \end{cases} \tag{6.7}$$

in agreement with Ref. 8. Nevertheless our approach is not equivalent to that of Bohr and Rand,[8] since the zeta function can be constructed even when the pressure cannot be defined (if, e.g., $\ln \|D_x T\|$ becomes infinite).

Indeed, the (possible) existence of the zeta function seems to us the best way to introduce the analog of the topological pressure when the function φ in (2.1) is not smooth on the invariant set J. Let us finally note that Benzi *et al.*[11] have found that the tent map also exhibits a phase transition in the pressure function.

7. CONCLUDING REMARKS

In this paper we have shown that for a large class of hyperbolic invariant sets, all the relevant dynamical variables can be extracted from the topological pressure. The reason is that, when the invariant measure satisfies certain uniformity properties and for a particular kind of partition of the set, the generalized dimensions, entropies, and Liapunov exponents can be expressed as partition functions whose thermodynamic limits are simply connected to the pressure. A special role is played by the generalized

Liapunov exponents. In fact, they are the easiest quantity to compute in numerical experiments: for example, in Refs. 11 the interested reader can find $L(q)$ curves for many dynamical systems, such as the Hénon map, the Hénon–Heiles model, and the Lorenz model. Moreover, Proposition 3 indicates that the $L(q)$ are related to the pressure function more directly than generalized entropies and dimensions.

Finally, by means of the Ruelle zeta function, we have tried to extend the thermodynamic formalism to nonhyperbolic invariant sets, provided that they are the closure of the fixed points of the map, but a phase transition could occur in this case.

APPENDIX A

Let us prove Proposition 1. Since $\log |T'(x)|$ is uniformly continuous on $T^{-1}[0, 1]$, for fixed $\varepsilon > 0$ there is $\chi_\varepsilon > 0$ such that for each pair of points $(x, y) \in T^{-1}[0, 1]$ with $|x - y| < \chi_\varepsilon$ we have $|T'(x)| \leqslant |T'(y)|\, e^\varepsilon$. Choose m so large that diameter $\mathscr{A}^{(m)} < \chi_\varepsilon$, put $\mathscr{A}^{(m)} = \mathscr{B}^{(0)}$, and consider the dynamical partitions starting from $\mathscr{B}^{(0)}$ [this obviously does not change the limit (2.11)]. Now, let us recall[15,16] that the uniform Gibbs measure μ_U is equivalent, with continuous Radon–Nikodyn derivative, to a probability "conformal" measure ν_c, which has the property that, for every measurable set B where T is injective, it satisfies

$$\nu_c(TB) = \int_B |T'(x)|^{D_{\mathrm{H}}}\, d\nu_c(x) \tag{A.1}$$

Applying iteratively this relation to the elements

$$B_\alpha^{(n)} \in \mathscr{B}^{(n-1)} = \bigvee_{k=0}^{n-1} T^{-k} \mathscr{B}^{(0)}$$

we get (note that for economy of notation we have put $B_\alpha^{(n)}$ instead of $B_\alpha^{(n-1)}$ as in the preceding sections)

$$\begin{aligned} &\psi_1 \mu_U(B_\alpha^{(n)}) \prod_{i=0}^{n-1} \min_{x_i \in T^i B_\alpha^{(n)}} |T'(x_i)|^{D_{\mathrm{H}}} \\ &\leqslant \mu_U(T^n B_\alpha^{(n)}) \\ &\leqslant \psi_2 \mu_U(B_\alpha^{(n)} \prod_{i=0}^{n-1} \max_{x_i \in T^i B_\alpha^{(n)}} |T'(x_i)|^{D_{\mathrm{H}}} \end{aligned} \tag{A.2}$$

where ψ_1 and ψ_2 are positive, finite constants deriving from the equivalence with the conformal measure. Using now the uniform continuity, we get

$$\begin{aligned} &\psi_1 \mu_U(B_\alpha^{(n)})\, e^{-n\varepsilon}\, |(T^n)'(x)|^{D_{\mathrm{H}}} \\ &\leqslant \mu_U(T^n B_\alpha^{(n)}) \\ &\leqslant \psi_2 e^{n\varepsilon} \mu_U(B_\alpha^{(n)})\, |(T^n)'(x)|^{D_{\mathrm{H}}} \end{aligned} \tag{A.3}$$

where x is any point in $B_\alpha^{(n)}$.

Since T^n is a strictly monotone of $B_\alpha^{(n)}$ onto $[0, 1]$, calling $W_{n,\alpha}$ its inverse and using Lagrange's theorem, we obtain

$$\delta(B_\alpha^{(n)}) = |W_{n,\alpha}(1) - W_{n,\alpha}(0)| = |(T^n)'(\xi)|^{-1} \tag{A.4}$$

where ξ belongs to $\mathrm{Int}(B_\alpha^{(n)})$ and $\delta(B^{(n)})$ is the diameter of $B_\alpha^{(n)}$. Choosing $x = \xi$ in (A.3), we finally have

$$\begin{aligned} &\psi_2^{-1} e^{-n\varepsilon}\, \delta(B_\alpha^{(n)})^{D_{\mathrm{H}}}\, \mu_U(T^n B_\alpha^{(n)}) \\ &\leqslant \mu_U(B_\alpha^{(n)}) \leqslant \psi_1^{-1} e^{n\varepsilon}\, \delta(B_\alpha^{(n)})^{D_{\mathrm{H}}}\, \mu_U(T^n B_\alpha^{(n)}) \end{aligned} \tag{A.5}$$

where $\mu_U(T^n B_\alpha^{(n)})$ is the finite measure of an element of the partition $\mathscr{B}^{(0)}$. Thus, we can bound the partition function (2.10) as (we suppose $q > 0$; for $q < 0$ similar bounds apply; besides, we neglect terms that remain finite in the limit $n \to +\infty$):

$$\begin{aligned} -n\varepsilon + \log \sum_{B_\alpha^{(n)}} \delta(B_\alpha^{(n)})^{qD_{\mathrm{H}}} &\leqslant (1-q)\, \mathscr{F}_{\mu_U}(q, n) \\ &\leqslant n\varepsilon + \log \sum_{B_\alpha^{(n)}} \delta(B_\alpha^{(n)})^{qD_{\mathrm{H}}} \end{aligned} \tag{A.6}$$

Taking the limit for $n \to \infty$, by (3.3) and the arbitrariness of ε, we get Proposition 1.

APPENDIX B

Let us prove Proposition 4. We use here the topological definition of the pressure and the relative notation given in Section 2. We consider the partition $\mathscr{A}^{(0)} = \{A_1, A_2, A_3, A_4\}$ of Λ with the closed set A_i shown in Fig. 3a. To be correct, we should consider an open cover of Λ, but our choice does not affect the final result. The partition $\mathscr{A}^{(n-1)} = \bigvee_{k=0}^{n-1} T^{-k} \mathscr{A}^{(0)}$ is done by two vertical strips intersecting the x axis in the intervals $[0, \gamma_a]$ and $[1/2, 1/2 + \gamma_b]$ and dividing each into 2^n rectangles; we show in Fig. 3b the partition $\mathscr{A}^{(1)}$, $n = 2$. Let us now concen-

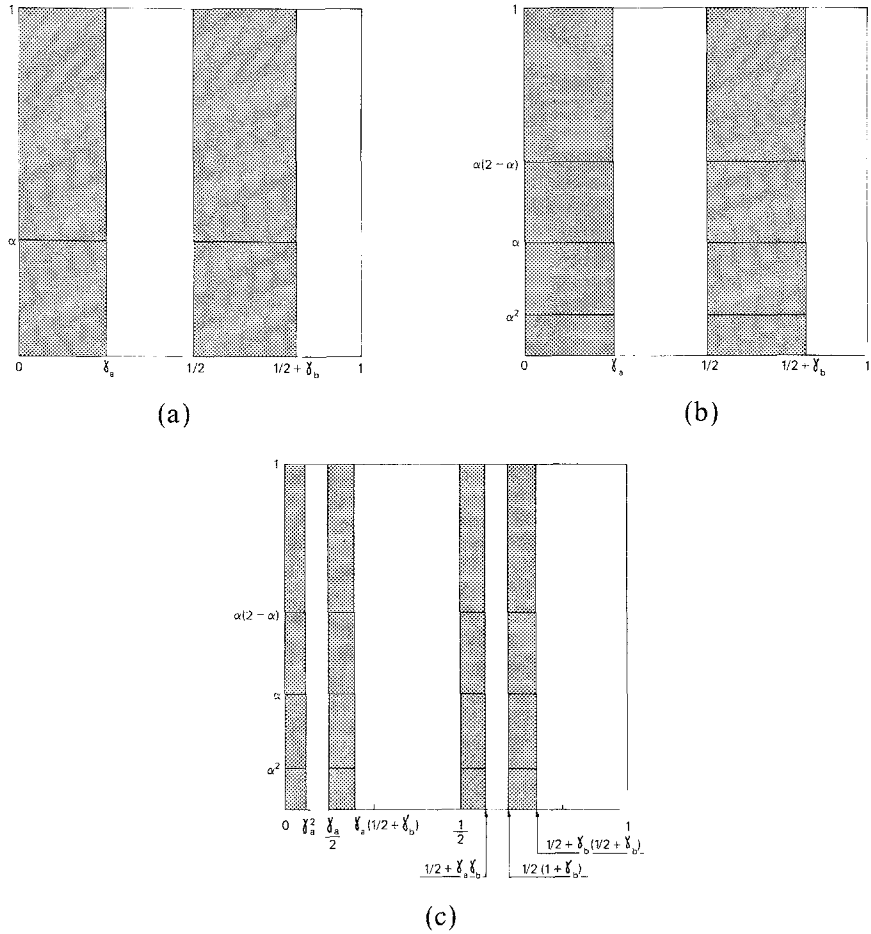

Fig. 3. Construction of the partition for the attractor of Baker's transformation: (a) $\mathscr{A}^{(0)}=\{A_1, A_2, A_3, A_4\}$, (b) $\mathscr{A}^{(1)}=\mathscr{A}^{(0)} \vee T^{-1}\mathscr{A}^{(0)}$, (c) $\overline{\mathscr{A}^{(0)}_{(1)}}=(\mathscr{A}^{(0)} \vee T^{-1}\mathscr{A}^{(0)}) \vee (\mathscr{A}^{(0)} \vee T\mathscr{A}^{(0)})$. Clearly, to get a partition of A, we have to intersect the shaded regions with A itself.

trate on one of the two strips: if we pick a point x in the rectangle $A_i \in \mathscr{A}^{(n-1)}$, $i=1,\dots,2^n$, belonging to the first strip, setting $1/\alpha=\chi_1$ and $1/(1-\alpha)=\chi_2$, we have

$$\widehat{D_x T_U^n} = \prod_{k=0}^{n-1} \chi_{i_k}$$

with $\{i_k\}$ a particular sequence of the two symbols $[1, 2]$. We obtain all the possible 2^n sequences by taking x in the different A_i of the first strip. By the definition of the pressure (2.4) and since we have two strips, we thus get

$$Z_n(q, \mathscr{A}^{(0)}) = 2 \sum_{i_1} \cdots \sum_{i_k} \prod_{k=0}^{n-1} \chi_{i_k}^{-q} = 2[\alpha^q + (1-\alpha)^q]^n \tag{B.1}$$

from which

$$P(q, \mathscr{A}^{(0)}) = \log[\alpha^q + (1-\alpha)^q] \tag{B.2}$$

The partition $\mathscr{A}^{(0)}$ is clearly not generating; then we choose another, finer partitin $\overline{\mathscr{A}^{(0)}_{(m-1)}}$ according to

$$\overline{\mathscr{A}^{(0)}_{(m-1)}} = \left(\bigvee_{k=0}^{m-1} T^{-k} \mathscr{A}^{(0)} \right) \vee \left(\bigvee_{k=0}^{m-1} T^{k} \mathscr{A}^{(0)} \right) \tag{B.3}$$

in such a way as to make the widths of the vertical strips smaller; see Fig. 3c. Repeating the above arguments, it is easy to see that

$$Z_n(q, \overline{\mathscr{A}^{(0)}_{(m-1)}}) = 2^{2m}[\alpha^q + (1-\alpha)^q]^n \tag{B.4}$$

from which the pressure of $\overline{\mathscr{A}^{(0)}_{(m-1)}}$ is the same as that of $\mathscr{A}^{(0)}$, independent of m. But diam $\overline{\mathscr{A}^{(0)}_{(m-1)}} \to 0$ for $m \to +\infty$, and we thus have

$$P(q) = \log[\alpha^q + (1-\alpha)^q] \tag{B.5}$$

We can now prove Proposition 5. The elements of the partition considered in Section 4 (which is subordinate to the unstable foliation) belong, for every $m \geqslant 1$, to $\overline{\mathscr{A}^{(0)}_{(m-1)}} \cap W^U(x)$, $x \in \Lambda$, and are simply the vertical pieces of the attractor inside each rectangle $\mathscr{R}^{(0)}_{(m-1)}$ of $\overline{\mathscr{A}^{(0)}_{(m-1)}}$.

The partition

$$\mathscr{A}^{(n)} = \left\{ \left(\bigvee_{k=0}^{m} T^{-k} \overline{\mathscr{A}^{(0)}_{(m-1)}} \right) \cap W^U(x) \right\}$$

is a refinement of $\mathscr{A}^{(n-1)}$, while the lengths of the horizontal sides of the rectangles $\mathscr{R}^{(n)}_{(m-1)} \in \bigvee_{k=0}^{n} T^{-k} \overline{\mathscr{A}^{(0)}_{(m-1)}}$ do not change under the backward iteration. The physical measure of such a rectangle is then the product of the conditional measure of the vertical side, which is equal to its length, and the conditional measure of the horizontal side along the stable direction, which depends only on $\overline{\mathscr{A}^{(0)}_{(m-1)}}$ and is thus bounded by two positive constants, say $\bar{\mu}^{(m)}_{\mathrm{MIN}}$ and $\bar{\mu}^{(m)}_{\mathrm{MAX}}$. Moreover, it is easy to verify that the vertical length of any rectangle is exactly equal to $(\widehat{D_x T_U^{(n+1)}})^{-1}$, x belonging to the same rectangle [see the expression for the "expanding" Jacobian given in (B.1)]. Then we have for the Renyi entropies (a similar argument extends to the generalized Liapunov exponents)

$$\frac{1}{n+1}\log\left[(\bar{\mu}^{(m)}_{\mathrm{MIN}})^q\sum(\widehat{D_x T^{n+1}_U})^{-q}\right]$$
$$\leqslant \frac{1}{n+1}\log\sum\mu(R^{(n)}_{(m-1)})^q$$
$$\leqslant \frac{1}{n+1}\log\left[(\bar{\mu}^{(m)}_{\mathrm{MAX}})^q\sum(\widehat{D_x T^{n+1}_U})^{-q}\right] \tag{B.6}$$

where the sum is over the rectangles $R^{(n)}_{(m-1)}$ and q is chosen positive, without any restriction. Taking the limit for $n\to+\infty$, we obtain that the q-order entropy of the partition $\mathscr{A}^{(0)}_{(m-1)}$ is just $(1-q)^{-1}P(q)$. We can then make the partition $\mathscr{A}^{(0)}_{(m-1)}$—hence the measure of the horizontal side of the rectangles—finer and finer as $m\to+\infty$: the uniform boundedness of the partition function (B.6) continues to hold—for every m—with different, but fixed constants $\bar{\mu}^{(m)}_{\mathrm{MIN}}$ and $\bar{\mu}^{(m)}_{\mathrm{MAX}}$. Therefore, the thermodynamic limit is the same, independent of m. This proves (4.2).

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REFERENCES

1. M. J. Feigenbaum, *J. Stat. Phys.* **46**:919 (1987).
2. P. Collet, J. L. Lebowitz, and A. Porzio, *J. Stat. Phys.* **47**:609 (1987).
3. G. Turchetti and S. Vaienti, in *Proceedings Conference in Honour of Prof. Ashour Cairo* (1987); *Phys. Lett. A* (1988), in press.
4. S. Vaienti, *J. Phys. A*, to appear.
5. R. Bowen, in *Lecture Notes in Mathematics*, No. 470 (1970).
6. D. Ruelle, *Thermodynamic Formalism* (Addison-Wesley, 1978).
7. P. Walters, *Am. J. Math.* **97**:937 (1975).
8. T. Bohr and D. Rand, *Physica* **25D**:387 (1987).
9. P. Grassberger, *Phys. Lett.* **97A**:227 (1983); R. Benzi, G. Paladin, G. Parisi, and A. Vulpiani, *J. Phys. A* **17**:3521 (1984); T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Shraiman, *Phys. Rev. A* **33**:1141 (1986).
10. P. Grassberger and I. Procaccia, *Phys. Rev. A* **31**:1872 (1985).

11. H. Fujsaka, *Progr. Theor. Phys.* **70**:1264 (1983); R. Benzi, G. Paladin, G. Parisi, and A. Vulpiani, *J. Phys. A* **18**:2157 (1985).
12. G. Benettin, L. Galgani, and J. M. Strelcyn, *Phys. Rev. A* **14**:2338 (1976).
13. M. Ljubich, *Ergod. Theory Dynam. Syst.* **3**:351 (1983).
14. M. F. Bernsley and S. Demko, *Proc. R. Soc. Lond. A* **399**:243 (1985).
15. D. Ruelle, *Ergod. Theory Dynam. Syst.* **2**:99 (1982).
16. D. Sullivan, in *Lecture Notes in Mathematics*, No. 1007 (1983).
17. P. Walters, *Trans. Am. Math. Soc.* **236**:121 (1978).
18. L. P. Kadanoff and C. Tang, *Proc. Natl. Acad. Sci. USA* **81**:1276 (1984).
19. I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinai, *Ergodic Theory* (Springer-Verlag, 1980).
20. G. Paladin and A. Vulpiani, *Lett. Nuovo Cimento* **41**:82 (1984); R. Badii and A. Politi, *Phys. Rev. Lett.* **52**:1661 (1984).
21. G. Paladin and A. Vulpiani, *J. Phys. A* **19L**:997 (1986).
22. G. Paladin and A. Vulpiani, *J. Phys. A*, in press.
23. A. Manning, *Ann. Math.* **119**:425 (1984).
24. F. Ledrappier and M. Misiurewicz, *Ergod. Theory Dynam. Syst.* **5**:595 (1985).
25. L. S. Young, *Ergod. Theory Dynam. Syst.* **2**:109 (1982).
26. H. McCluskey and A. Manning, *Ergodic Theory Dynam. Syst.* **3**:251 (1983).
27. Ya. B. Pesin, *Ergodic Theory Dynam. Syst.* **4**:405 (1984).
28. F. Ledrappier and L. S. Young, *Ann. Math.* **122**:509 (1985).
29. H. Brolin, *Ark.* Math. **6**:103 (1965).
30. H. G. E. Hentschel and I. Procaccia, *Physica* **8D**:435 (1983).
31. D. Ruelle, in *Lecture Notes in Physics*, No. 80.
32. D. A. Russel, J. D. Hanson, and E. Ott, *Phys. Rev. Lett.* **45**:1175 (1980).
33. J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields* (Springer-Verlag, 1983).
34. P. Grassberger, *Phys. Lett.* **97A**:227 (1983).
35. K. J. Falconer, *The Geometry of Fractal Sets* (Cambridge University Press, 1985).
36. P. Walters, *Trans. Am. Math. Soc.* **236**:121 (1978).
37. G. Servizi, G. Turchetti, and S. Vaienti, *Nuovo Cimento*, in press.
38. S. Vaienti, *J. Phys. A*, in press.